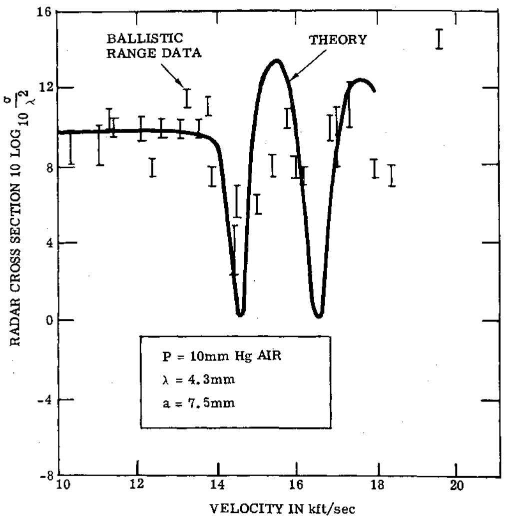

Fig. 2 Variation of radar cross section with velocity for a 7.5-mm radius sphere at 10-mm-Hg pressure; comparison of theory and experiment.

7.5-mm radius sphere at 10 mm Hg as seen by a 4.3-mm wavelength radar. The same kinds of comparisons have also been made for some previously published Canadian Armament Research and Development Establishment (CARDE) data, and similar results have been obtained. In all cases, it is evident that the theory is capable of predicting a very large decrease in cross section for these very thin plasma layers, something which was not possible with previous theories.

The details of radar cross section variation with speed depend critically on the development of the ionization around the body. In particular, we find that the results are very sensitive to the rate constants used in the nonequilibrium flow field calculations. Thus, there is a need for very good flow field calculations before exact comparisons between the theoretical and experimental results can be made. Work is continuing along the lines of improving the diffraction theory used in the prediction of the radar cross sections of plasma-covered metal bodies and in the calculation of the flow fields around blunt bodies. In conclusion, it appears that a theoretical explanation has been found for the large decrease in the radar cross section of a blunt-metal body when it is covered by a very thin plasma sheath. The decrease depends on partial coverage of the body by the plasma layer.

It is the angular gradients of the plasma properties around the body that cause severe diffraction of the radar wave. The details of the effect depend so critically upon the distribution of ionization in the plasma sheath that the effect may be useful as a flow field diagnostic.

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Lagrange Multiplier Techniques in Structural Analysis

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Nomenclature

P_i	= external loads
X_j	= loads between elements (internal loads)
X, P	= column of these loads
$LX + P$	= 0 is the matrix expression of the equilibrium equations at nodes
$[F]$	= combined flexibility matrix, overlapped and summed
$\frac{1}{2}X'[F]X$	= strain energy of elements concerned
u	= nodal deflections (column)
λ	= Lagrange multipliers (column)
Δu	= departure from rigid body motion (column)
u_0	= nodal deflections for unit rigid-body motions (a column for each)
$u_0\lambda$	= denotes the actual rigid body motion (λ is again a column)
δ	= misfits between elements measured along the X (column)
$[K]$	= combined stiffness matrix, overlapped and summed
$\frac{1}{2}u'[K]u$	= strain energy of elements concerned
$[k]$	= a small stiffness to earth
$\frac{1}{2}\lambda'u_0'[k]u_0\lambda$	= energy stored in light springs to earth

Introduction

The Argyris methods[2] generate their matrices for solution or inversion by processes of matrix multiplication. They predominantly solve structures using forces as variables. Asplund gives the theory elegant academic form and extends it to use forces in one part of a structure and displacements in the other.[3, 4] The direct stiffness method of Turner[5] avoids matrix multiplications (except of element size) by placing the coefficients directly into the equations to be solved. The generalized stiffness solution of Jones[6] is also direct in this sense.

Dallison Method

The dual of the Jones method is the Dallison force method of 1953,[1] which should never have been forgotten. Dallison introduced element equilibrium conditions using Lagrange multipliers. If, instead, we introduce nodal equilibrium conditions using λ as Lagrange multipliers, we find that a λ_i equals the nodal displacement u_i in the direction of the applied load P_i, that is, the constant term in the equilibrium equation. This depends on work being expressible as $\Sigma u_i P_i$, as does the direct stiffness method (see Ref. 4, Sec. Ld).

A mixed Dallison method is clearly possible, in which part of the structure is analyzed using flexibilities, the other part using stiffnesses, while still retaining the characteristic of directness. The Dallison force method, using nodal equilibrium, may be expressed as

$$\begin{bmatrix} F & L' \\ L & 0 \end{bmatrix} \begin{bmatrix} X \\ u \end{bmatrix} = \begin{bmatrix} \delta \\ -P \end{bmatrix} \tag{1}$$

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and the corresponding mixed method as

$$\begin{bmatrix} F_{11} & L_1' & 0 \\ L_1 & -K_{11} & -K_{12} \\ 0 & -K_{21} & -K_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \delta \\ -P_1 \\ -P_2 \end{bmatrix} \tag{2}$$

where suffix 1 denotes the part of the structure connected by the elements in F_{11} (although the option exists of connecting these nodes by further elements known by their stiffnesses K_{11}), and suffix 2 denotes the nodes treated by direct stiffness. To prove (1) without invoking Lagrange multiplier theory, we expand and solve

$$X = F^{-1}(-L'u + \delta) \tag{3}$$

$$LX + P = 0 \tag{4}$$

This can be proved true if we let x denote the displacements corresponding to X, just as the u correspond to P. Then, by the generalized leverage ratio theorem ascribed to Clebsch (see Ref. 4, Sec. Me), (4) gives $x = -L'u$, and (3) follows. If u now were held constant, and a foreign stiffness matrix were subtracted from the zero partition of (1), the P would be increased accordingly, as if additional parallel elements had been introduced. Equations (2) introduce further nodes that again modify P_1 if the u are held constant. An example illustrates this.

The springs in Fig. 1 are assumed to be too short by amounts δ_R, δ_S, and δ_T. In parallel with the springs illustrated is another spring system, not shown, with stiffnesses k_{uu}, k_{uv}, k_{vv} to earth. The equations already lose the characteristic appearance of equations involving Lagrange multipliers:

$$\begin{array}{lllllllllll} f_R R & & & & & - u & & & = & \delta_R \\ & f_S S & & & & + u/2^{1/2} & - & v/2^{1/2} & = & \delta_S \\ & & f_T T & & & + u/2^{1/2} & + & v/2^{1/2} & = & \delta_T \\ -R & + S/2^{1/2} & + T/2^{1/2} & & & -k_{uu} u & - & k_{uv} v & = & -U \\ & - S/2^{1/2} & + T/2^{1/2} & & & -k_{uv} u & - & k_{vv} v & = & -V \end{array} \tag{5}$$

From the point of view of numerical efficiency it is reassuring to observe that every coefficient in these equations can be given a useful job because the zero partition is now filled with k terms. Also, it is reassuring to note that symmetry is retained. These equations are correct, as may be seen by giving the displacements prescribed values:

$$\left.\begin{aligned} R &= (u + \delta_R)/f_R \\ S &= \frac{-(u/2^{1/2}) + (v/2^{1/2}) + \delta_S}{f_S} \\ T &= \frac{-(u/2^{1/2}) - (v/2^{1/2}) + \delta_T}{f_T} \\ U &= R - (S/2^{1/2}) - (T/2^{1/2}) + k_{uu}\, u + k_{uv}\, v \\ V &= (S/2^{1/2}) - (T/2^{1/2}) + k_{uv}\, u + k_{vv} v \end{aligned}\right\} \tag{6}$$

McKenzie Method

A simpler extension to stiffness methods has been developed with R. D. McKenzie of Rolls-Royce which improves the numerical conditioning when a very stiff structure is lightly sprung to earth by small stiffnesses $[k]$. Let the unearthed structure have stiffnesses $[K]$ containing large terms. Normally one solves $[K + k]\, u = P$. Define u_0 as the nodal deflections due to a unit rigid body motion and write the deflections as $\lambda u_0 + \Delta u$. Because $[K]\, u_0 = 0$, the equations may be written

$$\begin{bmatrix} K + k & ku_0 \\ u_0'\, k & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \lambda \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix} \tag{7}$$

There can be as many λ and u_0 as there are troublesome rigid body motions, and this technique may be justified if double precision arithmetic can be avoided by introducing a few λ.

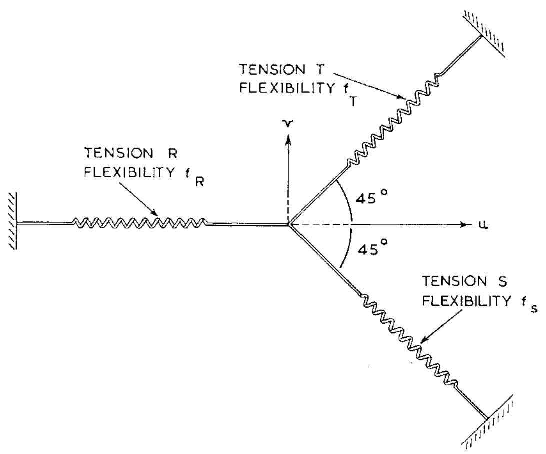

Fig. 1 Diagram of simple structure to illustrate the flexibility aspect of the Dallison mixed method.

A Generalization and a New Terminology

The Jones technique[5] is the dual of the basic Dallison method. The dual of the McKenzie technique would introduce certain combinations of the forces X_r as new variables, with the sole objective of improving the numerical conditioning. Thus, there are at least four distinct types of the so-called Lagrange multiplier. However, it is better at this stage to drop the term "Lagrange multiplier" altogether because, in every case, the additional variable has a clear physical meaning and because in (1), for example, it is impossible to say which of X and u is Lagrange multiplier and which is basic variable. It is a reciprocal relation most happily expressed by saying that X and u are "Lagrange opposite."

This change in terminology has far-reaching consequences. For a start, (7) could be applied to the u_2 group of (2), and, with four interrelated techniques available, the possibility arises of using them all in one problem, making a chain of unlimited length.

Equation Solving

With a direct method, especially the direct stiffness method, it is easy to save storage by considering substructures:

1) The structure can be divided up, and the internal nodal deflections, etc., for each substructure can be eliminated before final assembly.

2) The structure can be regarded as two substructures. One grows as elements are added until the other disappears, when the back substitution follows in reverse order. A fixed area is reserved in core, but the coefficients relate to different variables as the reduction proceeds. As each element is added, new variables are introduced and old ones are eliminated. At each elimination, the equation deleted is stored on tape, together with its code name and its position in the equation; these data are sufficient for the back substitution phase. Each deletion leaves room for a new variable.

Whichever technique is used, the large number of variables in the Dallison method no longer implies wasted storage and wasted arithmetic. True, it is more costly than the direct stiffness method, solved in the same way; but it would seem to compete with the conventional Argyris force methods regarding storage, arithmetic, simplicity, and development potential.

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A Solution of the Partial Differential Equation Governing Vibration of a Rod

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Introduction

In this paper, a numerical solution by the method of characteristics is presented for the transient response of a semi-infinite circular rod where one end is restrained radially and subjected to a step-axial pressure. The equation of motion for such a rod is based upon a one-dimensional theory of compressional waves including effects of the radial inertia and the radial shear derived by Mindlin and Herrmann.[1] It is well known that the result from this theory matches quite closely that of the exact theory due to Pochhammer. Several authors[2,3] have solved this problem by the method of integral transforms. Since this kind of equation of motion is composed of a number of simultaneous equations, the characteristic equation is a high-order polynomial, and the inversion integrals are complicated. Numerical methods must be used to evaluate those complicated inversion integrals. This suggests a direct numerical method to attack the problem. Since the system of equations is hyperbolic[4] and contains only one spatial variable, a simple numerical procedure by the method of characteristics can be used. Leonard and Budiansky[5] and Plass[6] applied the method of characteristics to the solution of the beam vibration problem by the Timoshenko theory. The author[7] applied it to solve the transient response of a thin-walled cylindrical tube under internal moving pressure.

Equation and Method of Solution

The homogeneous equations of motion and the relationship between the axial stress and displacement according to the Mindlin-Herrmann theory are as follows:

$$\varkappa^2 a^2 \mu(\partial^2 u/\partial x^2) - 8\varkappa_1^2(\lambda + \mu)u - 4\lambda\varkappa_1 a(\partial w/\partial x) = \rho a^2(\partial^2 u/\partial t^2) \quad (1)$$

$$2\lambda\varkappa_1 a(\partial u/\partial x) + (\lambda + 2\mu)a^2(\partial^2 w/\partial x^2) = \rho a^2(\partial^2 w/\partial t^2) \quad (2)$$

$$2P_x = 2\varkappa_1 a\lambda u + (\lambda + 2\mu)a^2(\partial w/\partial x) \quad (3)$$

in which u, w are the radial and axial displacements, respectively; P_x is the resultant axial stress on a section perpendicular to the axis of the rod; a is the radius of the rod; λ, μ are Lamé constants; ρ is the density of the rod; x is the axial distance; and t is the time variable. $\varkappa$ and $\varkappa_1$ are correction factors determined as follows: $\varkappa^2$ is the root of the equation

$$4[(1 - \zeta\varkappa^2)(1 - \varkappa^2)]^{1/2} = (2 - \varkappa^2)^2 \qquad 0 < \varkappa < 1 \quad (4)$$

where $\zeta = (1 - 2\nu)/2(1 - \nu)$, and ν is Poisson's ratio,

$$\varkappa_1^2 = \eta^2(\lambda + 2\mu)/8(\lambda + \mu) \quad (5)$$

where η is the lowest root of the equation

$$[(\lambda + 2\mu)/2\mu]\eta J_0(\eta) = J_1(\eta) \quad (6)$$

and J_0, J_1 are Bessel functions of order zero and one, respectively.

It is assumed that there is no motion initially for the rod and that a step axial pressure P_0 is applied for $t > 0$ at the end $x = 0$ while the rod is restrained radially at this end. The stress and the displacement at the end $x = \infty$ are not affected.

The system of equations (1) and (2) is hyperbolic,[4] and its characteristics are given by $dt/dx = \pm 1/c_1$; $dt/dx = \pm 1/c_2$, where $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$, the dilatational wave velocity; and $c_2 = \varkappa(\mu/\rho)^{1/2}$, the modified shear-wave velocity. If four dependent variables, $U_x = \partial U/\partial X$, $U_T = \partial U/\partial T$, $W_x = \partial W/\partial X$, and $W_T = \partial W/\partial T$, in which the dimensionless variables $U = u/a$, $W = w/a$, $X = x/a$, and $T = c_1 t/a$ are introduced, Eqs. (1) and (2) can be transformed into a set of four first-order equations in dimensionless form defined along the characteristics as follows.

Along the characteristics $dT/dX = \pm 1$,

$$dW_x \mp dW_T \pm \beta U_x dT = 0 \quad (7a, b)$$

Along the characteristics $dT/dX = \pm 1/\delta$, where $\delta = c_2/c_1$,

$$\delta dU_x \mp dU_T \mp (\alpha U + 2\beta W_x)dT = 0 \quad (8a, b)$$

In Eqs. (8a, b) U† can be expressed by the integral

$$U = U_0 + \int_{T_0}^{T} U_T dT \quad (9)$$

where U_0 is the radial displacement of the same point at time T_0. The constants in Eqs. (7a, b) and (8a, b) are

$$\alpha = 4\varkappa_1^2/(1 - \nu) \qquad \beta = 2\varkappa_1[\nu/(1 - \nu)]$$

Equation (3) can also be transformed to the dimensionless form by

$$P = \beta U + W_x \quad (10)$$

where

$$P = [2/a^2(\lambda + 2\mu)]P_x$$

There are two kinds of waves in this elastic system: the dilatational wave is propagated with dimensionless velocities

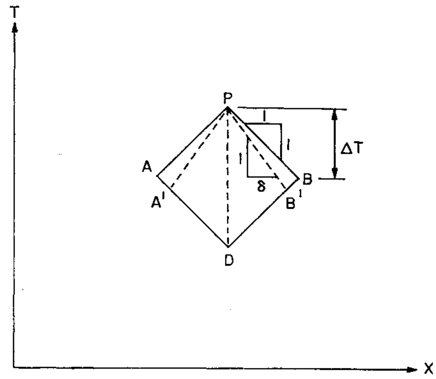

Fig. 1 Typical element of grid.

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† U may be treated as the fifth dependent variable; then the fifth equation is $U_T = \partial U/\partial T$. The final result will be the same.